How do Tom and Jerry play? A Simple Application of Convex Geometry in Game Theory

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ASSA 2024

Jan 5, 2024

Motivation

- Tom is a cat. Jerry is a mouse.
- They're playing hide-and-seek games in the house every day.

Motivation

- Suppose Tom and Jerry live in the same space. Tom and Jerry simultaneously choose their own location.
- Tom: get close to Jerry
- Jerry: get away from Tom
- What strategies should they play?

Applications

Hide-and-seek game

Feature: interest conflict between proximity and distance

- **Economy world**
  - immitator and innovator: design of products
- **Politics**
  - conservatives and radicals
- **Society**
  - police and criminal: allocation of resources in city management
- **Military**
  - information seeker and hider
  - attacker and defenser
- **Animal world**
  - predators and preys: distribution pattern of habitat
  - pest control: distribution pattern of pesticide and pests
- ......

Question: the characterization of equilibrium behavior?

- **Hide-and-seek game**
  - von Neumann (1953): one player aims to win by matching the other's decision, while the other aims to win by mismatching.
  - Fristedt (1977): hider hides a particle in $\mathbb{R}$ and seeker searches for it with a limited speed.
  - Kikuta (1990): hider hides in one of the $(n+1)$ cells and seeker searches for it with costs.
  - Petrosjan (1993): point-choosing model in a $\mathbb{R}^2$ triangle.
  - (Crawford and Iriberri, 2007, among many others): experiment of the hide-and-seek game.
  - Alpern (2008): hide-and-seek game in a network.

Literature review

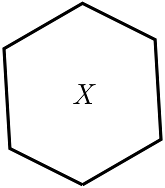
Games in a space

- **Hotelling model**
  - Hotelling (1929): the impact of location on duopoly competition.
  - d'Aspremont et al. (1979), Salop (1979), Owen and Shapley (1989), Mazalov and Sakaguchi (2003): variants of Hotelling model.
- **Matching pennies game**
  - Jordan (1993): proposed an example of 3-player matching pennies game.
  - McCabe et al. (2000): experiments about 3-player matching pennies games.
  - Goeree et al. (2003): risk averse behavior in generalized matching pennies games.
  - Cao and Yang (2014), Cao et al. (2019): the extension of matching pennies game in networks.
  - Bhattacharya (2016): information design in a matching pennies game.

Our Model

$\Gamma(X, \|\cdot\|_2)$

- A is the seeker (distance minimizer) and B is the hider (distance maximizer).
- $X \subseteq \mathbb{R}^n$ is the territory.
- X is a compact convex set.



- Player i chooses $x_i \in X$ simultaneously.
  - pure strategy profile: $x_i \in X$
  - mixed strategy profile: $\sigma_i \in \Delta(X)$ (probability measure, assumed to be a Borel measure)
  - Support of σ_i: $\mathrm{Supp}(\sigma_i)$

Our Model

$\Gamma(X, \|\cdot\|_2)$

- Utility: $p-$norm distance

$$\|x\|_p \equiv \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \quad \forall x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$$

  - this paper assumes $p = 2$
    - seeker: $u_A(x_A, x_B) = -\|x_A - x_B\|_2$
    - hider: $u_B(x_A, x_B) = \|x_A - x_B\|_2$
- Expected utility:

$$U_A(\sigma_A, \sigma_B) = \mathbf{E}_{x_A \sim \sigma_A, x_B \sim \sigma_B}(u_A) = \int_{X \times X} -\|x_A - x_B\|_2 \; d\sigma_A \; d\sigma_B$$

$$U_B(\sigma_A, \sigma_B) = \mathbf{E}_{x_A \sim \sigma_A, x_B \sim \sigma_B}(u_B) = \int_{X \times X} \|x_A - x_B\|_2 \; d\sigma_A \; d\sigma_B$$

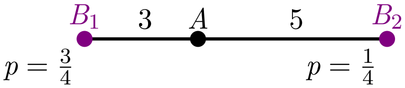
- Nash Equilibrium

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(x_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Delta(X), \quad \forall i \in \{A, B\}$$

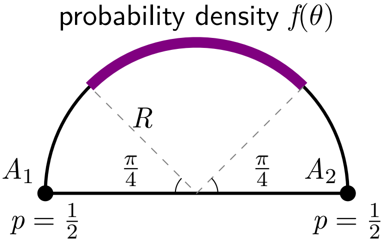
Two Examples

expected utility of hider (seeker) $= +(-)$ **expected distance**

- Black: seeker A
- Blue: hider B



$$U_B = \frac{3}{4} \times 3 + \frac{1}{4} \times 5$$



$$U_B = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2R\big(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\big) f(\theta)\, d\theta$$

Geometric definition

Ball and minimal cover ball

Definition

A ball in $\mathbb{R}^n$ is defined as the closed set

$$b(x, r) = \left\{x' \in \mathbb{R}^n \middle| \|x' - x\|_2 \leq r\right\}$$

where x is defined as the center of the ball and r is defined as the radius of the ball.

Definition

The ball $b(x^*, r^*)$ is a minimal cover ball of compact convex set X in $\mathbb{R}^n$ if the following two conditions are satisfied:

- $X \subseteq b(x^*, r^*)$.
- $\forall b(x, r)$ s.t. $X \subseteq b(x, r)$, there is $r^* \leq r$.

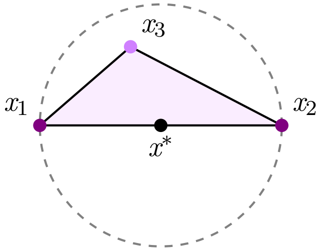
The minimal cover ball of compact convex set X is denoted as $b_{mc}(X)$.

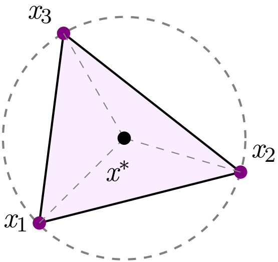
Geometric definition

Minimal cover ball

Examples: minimal cover ball

- obtuse triangle
- acute triangle





Geometric definition

Properties of minimal cover ball

Lemma

If X is a non-empty compact set in $\mathbb{R}^n$, then the minimal cover ball of X always exists and is unique.

Lemma

If X is a non-empty compact convex set in $\mathbb{R}^n$, then the minimal cover ball, denoted as $b_{mc}(X) = b(x^*, r^*)$, satisfies $x^* \in X$.

Main results

No pure strategy Nash Equilibrium

Proposition

In a Tom-and-Jerry Game $\Gamma\big(X, \|\cdot\|_2\big)$, in any Nash Equilibrium (if exists), the hider B's equilibrium strategy cannot be a pure strategy.

Corollary

In a Tom-and-Jerry Game $\Gamma\big(X, \|\cdot\|_2\big)$, *there exists no pure strategy Nash Equilibrium.*

Main results

Two types of Nash Equilibrium

Categorization:

- Type I strategy profile
  - seeker: pure strategy
  - hider: non-pure strategy
- Type II strategy profile
  - seeker: non-pure strategy
  - hider: non-pure strategy

Characterization Conditions

What should Type I equilibrium look like?

Support set condition

Suppose $\widehat{X}$ is a subset of X in $\mathbb{R}^n$. Player i's mixed strategy $\sigma_i(x_i)$ is said to "be supported by $\widehat{X}$", iff $\mathrm{Supp}(\sigma_i) \subseteq \widehat{X}$.

Center of mass condition

Suppose $\hat{x} \in \mathbb{R}^n$. Player i's mixed strategy $\sigma_i(x_i)$ is said to "have a center of mass at $\hat{x}$", iff $\mathbf{E}_{x_i \sim \sigma_i}(x_i) \equiv \int_X x_i \; d\sigma_i = \hat{x}$.

Theorem 1

Existence

In a Tom-and-Jerry Game $\Gamma\big(X, \|\cdot\|_2\big)$, suppose X is a compact convex set. Then there always exists a Type I Nash Equilibrium.

Characterization

A strategy profile (x_A^*, σ_B^*) is a Type I Nash Equilibrium, iff (x_A^*, σ_B^*) satisfies:

Theorem 1

1. The seeker A adopts a pure strategy $x_A^* = x^*$ at the center of the minimal cover ball of X, $b_{mc}(X) = b(x^*, r^*)$.
2. The hider B adopts a mixed strategy $\sigma_B^*(x_B)$ satisfying
   - support set condition: $\mathrm{Supp}(\sigma_B^*) \subseteq \partial X \cap \partial b_{mc}(X)$
   - center of mass condition: $\mathbf{E}_{x_B \sim \sigma_B^*}(x_B) = x^*$

In any Nash Equilibria of Tom-and-Jerry game, the equilibrium utility is

$$U_A^* = -r^*$$
$$U_B^* = r^*$$

Main results

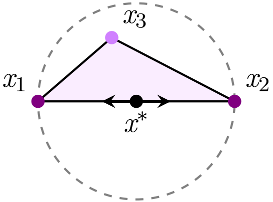
Intuition about Theorem 1

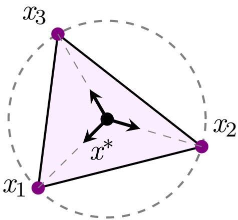
- the seeker A: the center of the minimal cover ball $b_{mc}(X)$
- the hider B: boundary of X and $b_{mc}(X)$: $\partial X \cap \partial b_{mc}(X)$
- the center of mass condition for the hider B

$$\mathbf{E}_{x_B \sim \sigma_B^*}(x_B) = \int_X x_B \; d\sigma_B^* = x^*$$

is equivalent to the best response condition that

$$\nabla_A U_A(x_A, \sigma_B^*)\big|_{x_A = x^*} = 0$$

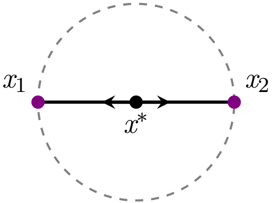




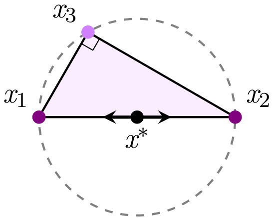
Main results

Intuition about Theorem 1

- Closed interval (segment) and right-angle triangle



(a) closed interval

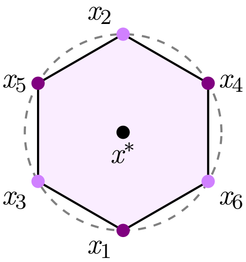


(b) right triangle

Main results

Intuition about Theorem 1

- Multiple Type I Nash Equilibria

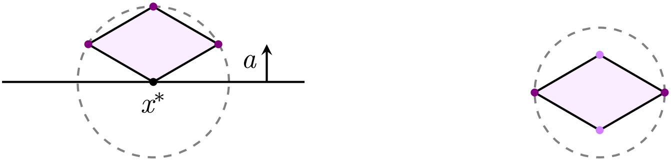


Main results

Intuition about Theorem 1

How to prove that there always exists a probabilistic distribution σ_B^* with $\mathrm{Supp}(\sigma_B^*) \subseteq \partial X \cap \partial b_{mc}(X)$ for the hider B, so that center of mass condition is satisfied?

- Observation
  - Not all the points in $\partial X \cap \partial b_{mc}(X)$ locate at the strictly same side of any hyperplane that passes through the center of minimal cover ball.
  - Otherwise: (a 2-dimension example)

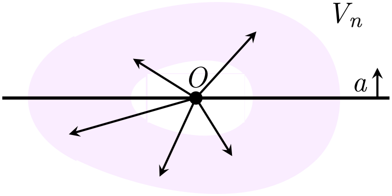


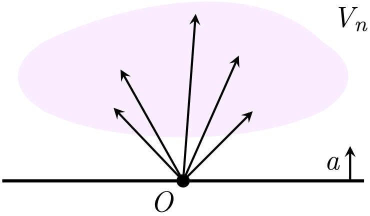
(The minimal cover ball can shift a little bit along a to strictly reduce the radius. This leads to contradiction!)

Main results

Intuition about Theorem 1 (Existence): an interesting observation

- Suppose V_n is a non-zero compact set in $\mathbb{R}^n$.
  - n is the number of space dimensions.
- The following two statements are exclusive:
  - For any hyperplane that passes through the origin, there always exist two vectors in V_n that lie on the different side of the hyperplane.
  - There exists a hyperplane a that passes through the origin, so that all the vectors in V_n lie strictly on the same side of the hyperplane.
- A 2-dimension example:

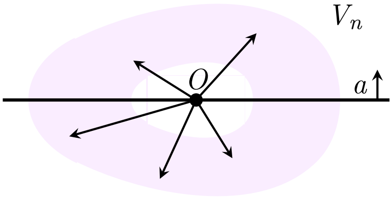


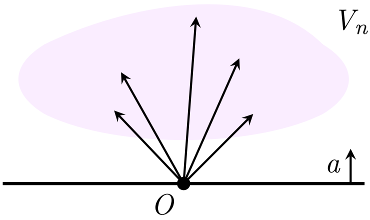


Intuition about Theorem 1

An interesting observation

- Correspondingly:
  - Able to find some vectors in V_n, whose non-trivial convex combination is 0
  - Any non-trivial convex combination of any vectors in V_n will never be 0





Question: How to express such a summation (or linear combination)?
———Borel measure and integration on a compact set.

Intuition about Theorem 1

An Extension of Farkas Lemma

An Extension of Farkas Lemma

For any $n \geq 1$, for any non-empty, non-zero and compact set V_n in $\mathbb{R}^n$, the following statement system

- A_n:

$$\exists\ \xi \in \mathcal{M}_+(V_n) - \{0\} \ \ s.t. \ \ \int_{V_n} v_n \ d\xi = 0$$

- B_n:

$$\exists\ a \in \mathbb{R}^n \ s.t. \ a^T v \geq 0 \ \forall v \in \text{Cone}(V_n), \ \text{``} =\text{''} \Leftrightarrow v = 0$$

- C_n:

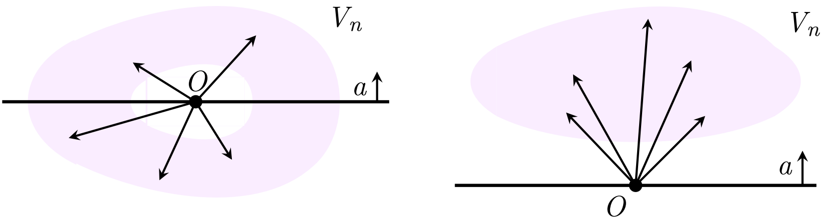
$$\exists\ a \in \text{Cone}(V_n) \ s.t. \ a^T v \geq 0 \ \forall v \in \text{Cone}(V_n), \ \text{``} =\text{''} \Leftrightarrow v = 0$$

Intuition about Theorem 1

An Extension of Farkas Lemma

satisfies the relationship

$$A_n \Leftrightarrow (\neg B_n) \Leftrightarrow (\neg C_n)$$



- Definition of measure space: relegated to Appendix
- Intuition of proof: properties of cone and dual cone

Proof of Theorem 1

Logics:

$$
\begin{aligned}
& b_{mc}(X) \text{ is the minimal cover ball} \\
\Rightarrow\quad & \text{For any hyperplane that passes through the center of the minimal cover ball, there always exists two vectors in } \partial X \cap \partial b_{mc}(X) \\
& \text{that are separated on (weakly) different sides of the plane} \\
\Leftrightarrow\quad & (\neg B_n) \text{ (Here } V_n = \left\{ x - x^* \middle| x \in \partial X \cap \partial b_{mc}(X) \right\}\text{)} \\
\Leftrightarrow\quad & A_n \\
\Leftrightarrow\quad & \exists\ \sigma_B^* \text{ s.t. } \int_{\partial X \cap \partial b_{mc}(X)} (x_B - x^*)\ d\sigma_B^* = 0 \\
\Leftrightarrow\quad & \exists\ \sigma_B^* \text{ satisfying center of mass condition } \mathbf{E}_{x_B \sim \sigma_B^*}(x_B) = x^*
\end{aligned}
$$

If X is compact and convex, then Type I Nash Equilibrium always exists.

Characterization Conditions

What should Type II equilibrium look like?

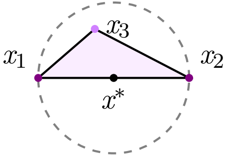
Intersection condition

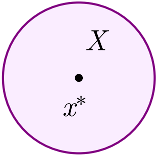
$\partial X \cap \partial b_{mc}(X) = \{x_1, x_2\}$, where $x_1 + x_2 = 2x^*$.

Extreme-point condition

Intersection condition holds and then

$$x_1, x_2 \notin \overline{\mathrm{EP}(X) - \{x_1, x_2\}}$$





Theorem 2

Existence

In a Tom-and-Jerry Game $\Gamma\big(X, \|\cdot\|_2\big)$, suppose the territory X is compact and convex. Then for the existence of Type II Nash Equilibrium,

- the intersection condition is a necessary condition
- the extreme-point condition is a sufficient condition

Characterization

A Type II strategy profile (σ_A^*, σ_B^*) is a Type II Nash Equilibrium, iff (σ_A^*, σ_B^*) satisfies:

Theorem 2

- The seeker A adopts a mixed strategy σ_A^* satisfying
  - $\mathrm{Supp}(\sigma_A^*) \subseteq \overline{x_1 x_2}$, where $\overline{x_1 x_2}$ is the unique diameter that is specified in the intersection condition.
  - $\mathbf{E}_{x_A \sim \sigma_A^*}(x_A) = x^*$
  - $\forall x_B \in X, \quad \mathbf{E}_{x_A \sim \sigma_A^*}\left(\|x_B - x_A\|_2\right) \leq r^*$, which disallows any possible deviation of the hider B.
- The hider B adopts a mixed strategy σ_B^* with equal probability weights only at the both endpoints of the diameter $\overline{x_1 x_2}$:

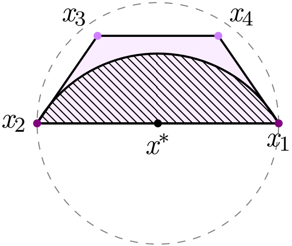
$$\sigma_B^*(x_B) = \begin{cases} \frac{1}{2}, & \text{if } x_B \in \{x_1, x_2\} \\ 0, & \text{if } x_B \notin \{x_1, x_2\} \end{cases}$$

Main results

Existence result

Proposition

For any compact convex set X_1 and X_2 with $X_1 \subseteq X_2$ and $b_{mc}(X_1) = b_{mc}(X_2)$, if the Tom-and-Jerry Game with common territory X_2 has a Type II Nash Equilibrium, then there also exists a Type II Nash Equilibrium in the Tom-and-Jerry Game with common territory X_1.



Main results

Existence result

- Is the extreme-point condition is a necessary and sufficient condition for the existence of Type II Nash Equilibrium?
    - **NO**.
- Are there some weird shapes that satisfy intersection condition but Type II Nash Equilibria do not exist?
    - **YES**.
    - We construct an example in $\mathbb{R}^2$ as follows, named "weird X".

Note:
Any compact convex polyhedron always satisfies the extreme-point condition.

Main results

Existence result

- Weird X
  - A "polyhedron" with infinite vertices and edges.
  - That is the reason why we leave the necessary and sufficient condition for the existence of Type II Nash Equilibrium as an open problem.

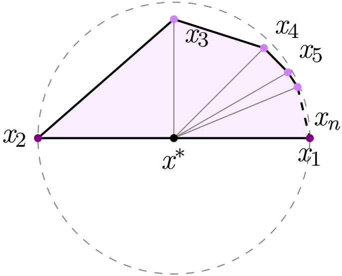


Figure: The shape of weird X.

Example

In $\mathbb{R}^2$ (expressed in cartesian coordinate), $x_1 = (r^*, 0), x_2 = (-r^*, 0)$ where $r^* \in \mathbb{R}_{++}$. Construct an infinite point sequence $\{x_n\}_{n=3}^{+\infty}$ converging to x_1

$$x_n = \left((1 - \frac{1}{2^n})r^* \cos\frac{\pi}{2(n-2)}, (1 - \frac{1}{2^n})r^* \sin\frac{\pi}{2(n-2)}\right), \quad n \geq 3$$

Define

$$X = \Big(\bigcup_{n=2}^{+\infty} \triangle x^* x_n x_{n+1}\Big) \cup \overline{x_1 x^*}$$

Here $\triangle$ means triangle and $\overline{x_1 x^*}$ means segment $x_1 x^*$.

Main results

Existence result

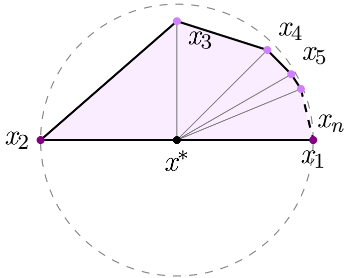


Figure: The shape of weird X. $\{x_n\}_{n=3}^{+\infty}$ is a point sequence converging to x_1.

We can prove:

- X is well-defined;
- X is compact and convex;
- X satisfies intersection condition;
- Type II Nash Equilibrium does not exist.

Summary

Type I and Type II Nash Equilibrium

- **Existence**

Type I	always exists
Type II	depends on the shape of X

- **Uniqueness**

$$\begin{array}{rl} & \text{Existence of Type } II \text{ Nash Equilibrium} \\ \Rightarrow & \text{a unique Type } I \text{ Nash Equilibrium (a point symmetric one)} \end{array}$$

- **Number of Type I and Type II Nash Equilibrium**

Type I \ Type II	0	continuum
1	✓ (rectangle)	✓ (segment)
continuum	✓ (ball)	✗

Table: The number of Type I and Type II Nash Equilibria when $X_A = X_B = X$ is compact and convex. ✓ means a possible combination and ✗ means an impossible one. For each possible combination, an example of the shape of X in $\mathbb{R}^2$ are given in the brackets.

Discussions

Minimal cover ball

Necessary and sufficient conditions of minimal cover ball

Suppose X is a compact convex set in $\mathbb{R}^n$ except for a singleton. $b(x^*, r^*)$ is a ball. Then $b(x^*, r^*)$ is a minimal cover ball of X iff the following three statements are all satisfied: (i) $X \subseteq b$; (ii) $|\partial X \cap \partial b| \geq 2$; (iii) there exists no $(n-1)-$dimension hyperplane passing through x^* so that all the points in $\partial X \cap \partial b$ lie strictly on the same side of the hyperplane, or formally,

$$\forall a \in \mathbb{R}^n, \quad \exists x_1 \neq x_2 \in \partial X \cap \partial b \quad s.t. \quad a^T(x_1 - x^*)\ a^T(x_2 - x^*) \leq 0$$

- condition (iii) $\Rightarrow \neg B_n$ in the geometric lemma system
- a part of the proof for Theorem 1

Discussions

The minimal cover ball and the corresponding convex optimization problem

Convex optimization problem

For any compact convex set X, construct a convex optimization problem

$$\min_{x\in\mathbb{R}^n}\max_{x'\in X}\|x-x'\|_p$$

where $p\geq 1$ is the norm distance. For the minimal cover ball of X (denoted as $b_{mc}(X)=b(x^*,r^*)$), its center is the optimal solution

$$x^*=\arg\min_{x\in\mathbb{R}^n}\max_{x'\in X}\|x-x'\|_p$$

and its radius is the optimal value

$$r^*=\min_{x\in\mathbb{R}^n}\max_{x'\in X}\|x-x'\|_p=\max_{x'\in X}\|x^*-x'\|_p$$

Discussions

Solution to the minimal cover ball for polyhedron

Definition

X is a compact convex polyhedron in $\mathbb{R}^n$, if and only if X is a compact set in $\mathbb{R}^n$ and there exist $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ so that $X = \left\{x \in \mathbb{R}^n \middle| Ax \leq b\right\}$.

Lemma

If X is a compact convex polyhedron and the minimal cover ball of X is $b_{mc}(X)$, then

$$\partial X \cap \partial b_{mc}(X) \subseteq \mathrm{EP}(X)$$

where $|\mathrm{EP}(X)| < +\infty$ is a finite set.

Extensions

Alternative settings

- Ball surface territory X
  - When X is a ball surface S^n
- Unoverlapped territory: $X_A \neq X_B$
  - Player i's territory: X_i
    - Partially overlapped territory
    - Complementary territory

Extensions

Ball surface territory

Assumption 2

$X = S^n$, where

$$S^n = \left\{(z_1, z_2, \cdots, z_{n+1}) \in \mathbb{R}^{n+1} \middle| \sum_{k=1}^{n+1} z_k^2 = 1\right\}$$

Angle distance metric on ball surface

For any two points x, y in a sphere S^n in $\mathbb{R}^{n+1}$, the distance between x and y, denoted as $\langle x, y\rangle$, is defined as the angle between x and y:

$$\langle x, y\rangle = \arccos(x^T y) \in [0, \pi] \quad \forall x, y \in S^n$$

- $U_A(\sigma_A, \sigma_B) = \mathbf{E}_{x_A \sim \sigma_A, x_B \sim \sigma_B}(-\langle x_A, x_B\rangle)$
- $U_B(\sigma_A, \sigma_B) = \mathbf{E}_{x_A \sim \sigma_A, x_B \sim \sigma_B}(\langle x_A, x_B\rangle)$

Extensions

Ball surface territory

Theorem 3

In a Tom-and-Jerry Game with ball surface territory and angle distance, there exists no Type I Nash Equilibrium.

Proof: If the equilibrium strategy of seeker is a single point, then this will lead to contradiction.

Extensions

Ball surface territory

Theorem 4

In a Tom-and-Jerry Game with ball surface territory and angle distance, the mixed strategy profile (σ_A^*, σ_B^*) is a mixed strategy Nash Equilibrium, if both players' strategy profiles satisfy point symmetry with regard to the center of the ball, i.e.

$$\forall x \in S^n, \quad \forall i \in \{A, B\}, \quad \sigma_i^*(x) = \sigma_i^*(-x)$$

In any mixed strategy Nash Equilibria, the equilibrium utility is

$$U_A^* = -\frac{\pi}{2}, \quad U_B^* = \frac{\pi}{2}$$

- Average angle distance: a quarter of a circle
- Equilibrium utility: independent of the space dimension n.

Extensions

Ball surface territory

- Theorem 4 is just a sufficient condition for a strategy profile being a mixed strategy Nash Equilibrium.
- Necessity?

Conjecture

In a Tom-and-Jerry Game with ball surface territory and angle distance, if a mixed strategy profile (σ_A^*, σ_B^*) is a mixed strategy Nash Equilibrium, then both players' strategy profiles satisfy point symmetry with regard to the center of the ball, i.e.

$$\forall x \in S^n, \quad \forall i \in \{A, B\}, \quad \sigma_i^*(x) = \sigma_i^*(-x)$$

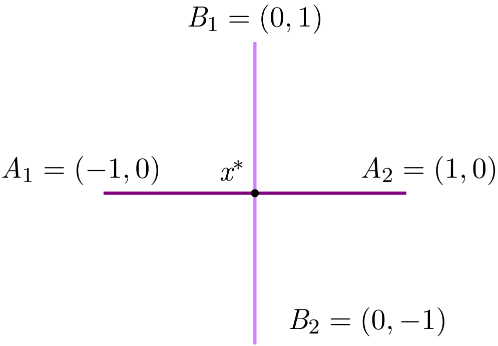
We leave the rigorous proof as an open problem.

Extensions

Partially overlapped territory

Example

X_A and X_B are two crossing segments in $\mathbb{R}^2$. All the Nash Equilibria in this example can be characterized as (x^*, σ_B^*), where $\sigma_B^*(B_1) = \lambda$ and $\sigma_B^*(B_2) = 1 - \lambda$ and $\lambda \in [0, 1]$.

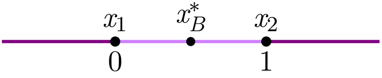


Extensions

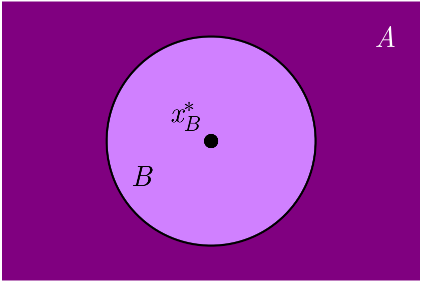
Complementary territory

Example

Assume that X_B is a compact convex set and $X_A = (\mathbb{R}^n - X_B) \cup \partial X_B$ (so that X_A is also a closed set).



(a)



(b)

Conclusion

This paper

Contributions

- interest conflict between proximity and distance
- characterization of Type I and Type II Nash Equilibrium
- a combination of game theory and convex analysis

Further exploration

- necessary and sufficient condition for the existence of Type II Nash Equilibrium for general cases
- ball surface territory X
- assumptions
  - territory assumptions to be relaxed
    - non-convex territory X
    - unoverlapped territory $X_A \neq X_B$
  - utility functions to be relaxed
    - any $p-$norm distance
    - non-linearly dependent on distance

References

Alpern, S. (2008). Hide-and-seek games on a tree to which eulerian networks are attached. *Networks*, 52(3):162–166.

Bhattacharya, S. (2016). Campaign rhetoric and the hide-and-seek game. *Social Choice and Welfare*, 47(3):697–727.

Cao, Z., Qin, C., Yang, X., and Zhang, B. (2019). Dynamic matching pennies on networks. *International Journal of Game Theory*, 48:887–920.

Cao, Z. and Yang, X. (2014). The fashion game: Network extension of matching pennies. *Theoretical Computer Science*, 540-541:169–181. Combinatorial Optimization: Theory of algorithms and Complexity.

Crawford, V. P. and Iriberri, N. (2007). Fatal attraction: Salience, naïveté, and sophistication in experimental "hide-and-seek" games. *American Economic Review*, 97(5):1731–1750.

d'Aspremont, C., Gabszewicz, J. J., and Thisse, J.-F. (1979). On hotelling's "stability in competition". *Econometrica*, 47(5):1145–1150.

References

Fristedt, B. (1977). Hide and seek in a subset of the real line. *International Journal of Game Theory*, 6:135–165.

Goeree, J. K., Holt, C. A., and Palfrey, T. R. (2003). Risk averse behavior in generalized matching pennies games. *Games and Economic Behavior*, 45(1):97–113. First World Congress of the Game Theory Society.

Hotelling, H. (1929). Stability in competition. *The Economic Journal*, 39(153):41–57.

Jordan, J. (1993). Three problems in learning mixed-strategy nash equilibria. *Games and Economic Behavior*, 5(3):368–386.

Kikuta, K. (1990). A hide and seek game with traveling cost. *Journal of the Operations Research Society of Japan*, 33(2):168–187.

Mazalov, V. and Sakaguchi, M. (2003). Location game on the plane. *International Game Theory Review*, 05(01):13–25.

References

McCabe, K. A., Mukherji, A., and Runkle, D. E. (2000). An experimental study of information and mixed-strategy play in the three-person matching-pennies game. *Economic Theory*, 15(2):421–462.

Owen, G. and Shapley, L. S. (1989). Optimal location of candidates in ideological space. *International Journal of Game Theory*, 18:339–356.

Petrosjan, L. A. (1993). *Differential Games of Pursuit*. WORLD SCIENTIFIC.

Salop, S. C. (1979). Monopolistic competition with outside goods. *The Bell Journal of Economics*, 10(1):141–156.

von Neumann, J. (1953). A certain zero-sum two-person game equivalent to the optimal assignment problem. In Kuhn, H. W. and Tucker, A. W., editors, *Contributions to the Theory of Games*, volume II of *Annals of Mathematics Studies (AM-28)*, pages 5–12. Princeton University Press.

Thank you!

Your comments will be highly appreciated.